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Topological sectors and gauge invariance in massive vector-tensor theories in D≥4

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Abstract

A family of locally equivalent models is considered. They can be taken as a generalization to $d+1$ dimensions of the Topological Massive and "Self-dual" models in 2+1 dimensions. The corresponding 3+1 models are analized in detail. It is shown that one model can be seen as a gauge fixed version of the other, and their space of classical solutions differs in a topological sector represented by the classical solutions of a pure BF model. The topological sector can be gauged out on cohomologically trivial base manifolds but on general settings it may be responsible of the difference in the long distance behaviour of the models. The presence of this topological sector appears explicitly in the partition function of the theories. The generalization of this models to higher dimensions is shown to be straightfoward.

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One of the motivations for studying field theories in 2+1 dimensions is that, being more tractable, one hopes to get some insight on their higher dimensional generalizations. This picture becomes more interesting when the lower dimensional models provide new ideas for the higher dimensional ones. This is the case of the so called "string" fractional statistics model [1] [2], which constitutes a generalization of the fractional statistics concept in 2+1 dimensions [3]. In the former example the role of the topological Chern-Simons term in 2+1 dimensions is generalized by an, also topological, BF term. In both cases the statistics appears as a manifestation of the topological structure of the base manifold.

The non-trivial topological nature of the base manifold may impose conditions on the equivalence between different physical models. In these situations, the possible global contributions of the topological terms to the observables of the theories may restrict their relation to hold on cohomological trivial sectors of the base manifold. This is the scheme between two different descriptions of massive spin 1 excitations in 2+1 dimensions: the "Self-dual" (SD) [4] and the Topological Massive (TM) models [5] [6]. On simply connected manifolds these two models are completely equivalent [7], and it can be shown that the SD model correspond to a gauge fixed version of the TM gauge theory [8]. Nevertheless, the space of solutions of both theories could be different. In fact, beside their common solutions there is a topological sector in the space of solutions of the TM model not present in the SD one. This topological sector is filled by all the flat connections on the base manifold [9]. This will not constitute any obstacle on simply connected manifolds, because this flat connections could be gauged out in the TM model. But on general settings, the gauge fixing procedure can only be performed locally, so the equivalence between both models will be conditioned to this level. This situation of global inequivalence persists if we use the usual Stuckelberg form of the SD model. Instead, to get a global relation between both models, we have to modify the SD action adding to the potential a_μ a closed but not necessarily exact 1-form ω_μ [9]. So, the global equivalence is obtained patching and sewing "SD formulations" over simply connected sectors of the base manifold. The so obtained modified SD action is gauge invariant and corresponds to a pure Chern-Simons model superposed on the original SD one [10]. As it could be expected, on simply connected sectors, the modified SD action turns to be the Stuckelberg form of the original one. It can also be shown, in a path integral approach, that the TM model can be obtained as a dualized version of the SD one [11].

In this letter we will show that this scheme of local and global equivalence between the SD and TM models, and their gauge fixing relation, can be generalized to higher space-time dimensions. We first study the generalization to 3+1 dimensions. The two models to be considered are well known and their comparision with the 2+1 picture has been noticed and used in different contexts [17][12]. It will be shown that one of the models can be taken locally as a gauge fixed version of the other. Also we will prove that on base manifolds, with a non-trivial topological structure, both models might have different long-distance behaviour. This difference, as in the 2+1 analogs, is due to a topological sector in the space of classical solutions which is not common between both models. This topological sector corresponds in d+1 dimensions to the classical solutions of a BF model. The presence of this sector is shown to appear in the partition function of the gauge invariant model. The generalization to d+1 dimensions is straihgtfoward through the formulation of both models in terms of the duals of the antisymmetric tensors.

In 3+1 dimensions massive spin 1 excitations can be described by the gauge invariant action [13]

$$S^4_{TM} = \int d^4x \Big[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{12\mu^2} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{4} \varepsilon^{\mu\nu\lambda\rho} B_{\mu\nu} F_{\lambda\rho} \Big] \,, \tag{1}$$

where $H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\lambda B_{\mu\nu} + \partial_\nu B_{\lambda\mu}$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ represent, respectively, the Kalb-Ramond and Maxwell field strengths. S^4_{TM} is invariant (up to a total divergence) under the gauge transformations $\delta B_{\mu\nu} = \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$, $\delta A_\mu = \partial_\mu \lambda$ and constitutes a generalization, to 3+1 dimensions, of the TM model [17]. In this model the two polarization states of the Maxwell field combine with the unique degree of freedom of the Kalb-Ramond field to produce a massive spin 1 excitation [13]$-$[17]. The equations of motion that arise from S^4_{TM} are

$$\partial_\nu F^{\nu\mu} - \frac{1}{2} \varepsilon^{\mu\nu\lambda\rho} \partial_\nu B_{\lambda\rho} = 0 \,, \tag{2}$$

$$\frac{1}{\mu^2} \partial_\lambda H^{\lambda\mu\nu} - \varepsilon^{\mu\nu\lambda\rho} \partial_\lambda A_\rho = 0 \,, \tag{3}$$

where we notice that closed forms $A = A_\mu dx^\mu$ and $B = B_{\mu\nu} dx^\mu \wedge dx^\nu$ (with $dA = 0$ and $dB = 0$) are always solutions of the system. The relation with the Proca theory is obtained by direct inspection: from its equation of motion, $\partial_\nu F^{\nu\mu} - \mu^2 A^\mu = 0$, we see that A_μ is transverse (or it is a co-closed 1-form), so it can be thought locally as the dual of an exact 3-form (or a co-exact 2-form); this is the second term in (2) and equation (3) ensures the identification. In other direction, the non-abelian generalization of this model, proposed by Freedman and Townsend [18], can be obtained from S^4_{TM} using the self-interaction mechanism [20].

The local relation between S^4_{TM} and the Proca model justify the comparison with the first order form of the latter [18] [19]

$$S^4_P = \int d^4x \Big[\frac{1}{4}\varepsilon^{\mu\nu\lambda\rho}B_{\mu\nu}F_{\lambda\rho} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{\mu^2}{2}A_\mu A^\mu\Big] \ , \tag{4}$$

which is, also, a first order form of the massive Kalb-Ramond model [17]−[19], albeit this model has a "spin jump" in the zero mass limit [21] [14] [15] [16] [19].

The equations of motion of S^4_P are

$$\frac{1}{2}\varepsilon^{\mu\nu\lambda\rho}\partial_\nu B_{\lambda\rho} - \mu^2 A_\mu \ = \ 0 \ , \tag{5}$$

$$\varepsilon^{\mu\nu\lambda\rho}\partial_\lambda A_\rho - B^{\mu\nu} \ = \ 0 \ , \tag{6}$$

where we observe that non-zero closed forms A and B do not belong to the space of solutions. So on general manifolds there would be a topological sector in the space of solutions of S^4_{TM} not present in the corresponding space of the model described by S^4_P. We recognize in S^4_P the generalization, to 3+1 dimensions, of the SD model.

The above mentioned models can be rewritten as

$$S^{4\,\star}_P = \int d^4x \Big[\frac{1}{2}T^{\mu\nu}F_{\mu\nu} + \frac{1}{4}T^{\mu\nu}T_{\mu\nu} - \frac{\mu^2}{2}A^\mu A_\mu\Big] \ , \tag{7}$$

and

$$S^{4\,\star}_{TM} = \int d^4x \Big[\frac{1}{2\mu^2}\partial_\mu T^{\mu\nu}\partial_\lambda T^\lambda{}_\nu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}T^{\mu\nu}F_{\mu\nu}\Big] \ , \tag{8}$$

where $T^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\lambda\rho}B_{\lambda\rho}$ are the components of ${}^\star B$. $S_{TM}{}^\star$ is invariant under the gauge transformations $\delta A_\mu = \partial_\mu\lambda$ and $\delta T^{\mu\nu} = \varepsilon^{\mu\nu\lambda\rho}\partial_\lambda\xi_\rho$. The topological sector is now filled by closed 1-forms A and co-closed 2-forms T, which are allways solutions of $S_{TM}{}^\star$. The generalization, to d+1 dimensions of S^4_P and S^4_{TM} is obtain directly from (7) and (8) if we use the identification $T = {}^\star B$ with B a $(d-1)$-form. We will keep then working in 3+1 dimensions and the results to higher dimensions are trivially generalized taking care of the identification.

Let us start showing the canonical equivalence of S^4_P and S^4_{TM} over a cohomological trivial region of space-time. We suppose that the base manifold is $M_4 = R \times \Sigma_3$, with Σ_3 a compact orientable 3-manifold. Starting with S^4_{TM}, after performing the canonical analysis we arrive to the hamiltonian density

$$\begin{aligned}\mathcal{H}^4_{TM} = \mu^2\Pi_{ij}\Pi_{ij} + \frac{1}{2}\Pi_i\Pi_i \ &+ \ \frac{1}{4}B_{ij}B_{ij} + \frac{1}{2}\varepsilon_{ijk}\Pi_i B_{jk} + \\ &+ \ \frac{1}{4}F_{ij}F_{ij} + \frac{1}{12\mu^2}H_{ijk}H_{ijk} \ ,\end{aligned} \tag{9}$$

subject to the first class constraints Θ_a

$$\theta = -\partial_i \Pi_i \quad , \tag{10}$$

$$\theta_i = -\partial_j \Pi_{ji} + \frac{1}{2}\varepsilon_{ijk}\partial_j A_k \; , \tag{11}$$

where Π_i and Π_{ij} are the conjugated momenta associated to A_i and B_{ij} (our metric signature is $(-+++)$). The non-canonical variables A_0 and B_{0i} appear as Lagrange multipliers associated to the constraints Θ_a. This set of constraints is reducible (because $\partial_i\theta_i = 0$) and implies the residual gauge invariance $\delta B_{0i} = -\partial_i \xi$.

Going to S_P^4, after eliminating A_0 and B_{0i}, we will arrive, taking the kinetic part as $\dot{B}_{ij}\varepsilon_{ijk}A_k$, to the hamiltonian density

$$\begin{aligned} \mathcal{H}_P^4 = \frac{\mu^2}{2}A_iA_i + \frac{1}{4}B_{ij}B_{ij} &+ \frac{1}{4}F_{ij}F_{ij} + \\ &+ \frac{1}{12\mu^2}H_{ijk}H_{ijk} \; , \end{aligned} \tag{12}$$

and the second class constraints Φ_A

$$\varphi_i = \Pi_i \quad , \tag{13}$$

$$\varphi_{ij} = \Pi_{ij} + \frac{1}{2}\varepsilon_{ijk}A_k \equiv \varepsilon_{ijk}\Psi_k \; . \tag{14}$$

The algebra of the constraints φ_i, Ψ_k has the only non-vanishing equal time Poisson brackets $\{\varphi_i(x), \Psi_j(y)\} = -(1/2)\delta_{ij}\delta^3(x-y)$. This allows us to take half of the constraints in (13,14) as first class constraints Θ_a, and the other half as gauge fixing conditions Υ_a [8] [24]. We take $\Theta_a = (-\partial_i\varphi_i, \varepsilon_{ijk}\partial_j\Psi_k) = (\theta, \theta_i^T)$, $\Upsilon_a = (-\partial_i\Psi_i, \varepsilon_{ijk}\partial_j\varphi_k) = (\Upsilon, \Upsilon_i^T)$. The bi-directional identification of the sets $\Phi_A \leftrightarrow (\Theta_a, \Upsilon_a) \equiv \phi_A$ is possible only on sectors where the first and second cohomology groups in Σ_3 are trivial, so the harmonic parts are taken to be zero. This division in first and second class constraints incite us to think on the underlying gauge theory. So we look for the gauge invariant hamiltonian [8]

$$\begin{aligned} \widetilde{H}_P^4 = H_P^4 &+ \int d^3x \Big[\Big[\alpha_a(x)\Theta_a(x) + \beta_a(x)\Upsilon_a(x)\Big] + \\ &+ \int d^3y \Big[\beta_{AB}(x,y)\phi_A(x)\phi_B(y)\Big]\Big] \quad , \end{aligned} \tag{15}$$

which differs from H_P^4 by combinations of the constraints, and satisfies homogeneous Poisson brackets with the defined first class constraints. Some of the coefficients, like the α's, will remain arbitrary. But there is a particular solution for wich we get $\widetilde{H}^4_P = H^4_{TM}$. This relation can be written explicitly as

$$
\begin{aligned}
\widetilde{H}^4_P &= H^4_P + \int d^3x \Big[\frac{1}{2}\varphi_i(\varphi_i + \varepsilon_{ijk}B_{jk}) + 2\mu^2\Psi_i(\Psi_i - A_i)\Big] \\
&\equiv H^4_{TM} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad .
\end{aligned}
\tag{16}
$$

If we go to the functional integral (the partition function), the measure [25] takes the form

$$
det\{\Phi_A, \Phi_B\}^{\frac{1}{2}}\delta(\Phi_A) = det\{\Theta_a, \Upsilon_b\}\delta(\theta)\delta(\theta^T_i)\delta(\Upsilon)\delta(\Upsilon^T_i) \ , \tag{17}
$$

and it can be shown that the right-hand side of this equation is the measure we would get in the functional integral of S^4_{TM} after reducing it to the independent physical modes [26]. In fact, in the process to obtain the effective, BRST invariant, action of S^4_{TM} we find that due to the reducibility property of θ_i there is a residual gauge invariance that must be fixed. This residual invariance comes from the arbitrariness in the longitudinal parts of not only B_{0i}, as we said, but also of the pair of ghost-antighost $(D_i, \overline{D}_i)$ accompanying θ_i and the Lagrange multiplier (E_i) associated with the gauge fixing constraint [26]. In order to fix these residual invariances in a BRST invariant way we must introduce triplets (ghost, antighost, multiplier) for each invariance. Let the triplet due to B_{0i} be $(d, \overline{d}, b)$ and the triplets due to D_i, $\overline{D}_i$ and E_i be respectively $(d_{\overline{a}}, \overline{d}_{\overline{a}}, b_{\overline{a}})$, with $\overline{a} = 1, 2, 3$. The non-null BRST transformation of the ghosts are $(\delta_{BRST}F = \zeta\hat{\delta}F$ with $\delta^2_{BRST}F = 0)$

$$
\begin{array}{cccc}
\hat{\delta}D_i = \partial_i d_1 & \hat{\delta}\overline{D}_i = -E_i + \partial_i d_2 & \hat{\delta}E_i = \partial_i d_3 & \hat{\delta}\overline{d}_{\overline{a}} = -b_{\overline{a}} \\
\hat{\delta}d_2 = d_3 & \hat{\delta}d = \dot{d}_1 & \hat{\delta}\overline{d} = -b &
\end{array} \quad . \tag{18}
$$

For A_μ and $B_{\mu\nu}$, the transformations are

$$
\hat{\delta}A_\mu = \partial_\mu C, \quad \hat{\delta}B_{ij} = \partial_i D_j - \partial_j D_i, \quad \hat{\delta}B_{0i} = \dot{D}_i - \partial_i d \ , \tag{19}
$$

where C is the ghost of the triplet $(C, \overline{C}, E)$ associated to the gauge invariance of A_i. The parity of the involved fields is clear from the context if we take account that $\hat{\delta}$ changes it. A good gauge fixing condition of these residual invariances results to be the cancellation of the projection of B_{0i}, D_i, $\overline{D}_i$ and E_i in its longitudinal parts, *i.e.*

$$
\Upsilon_D = \partial_i D_i, \qquad \Upsilon_{\overline{D}} = \partial_i \overline{D}_i, \qquad \Upsilon_E = \partial_i E_i, \qquad \Upsilon_B = \partial_i B_{0i} \ . \tag{20}
$$

The effective lagrangian will be [26] $\sim p\dot{q} - \mathcal{H}_{TM} - A_0\theta - B_{0i}\theta_i + \hat{\delta}(\overline{D}_{\mathcal{A}}\Upsilon_{\mathcal{A}})$, where $\overline{D}_{\mathcal{A}}$ and $\Upsilon_{\mathcal{A}}$ stands, respectively, for the antighosts (of *all* the triplets) that where introduced, and the corresponding gauge fixing conditions. p and q abreviate Π_i, Π_{ij} and A_i, B_{ij}, respectively. Now having all the gauge freedom fixed we go to the functional integral and start its reduction to the genuine physical modes. For this, we integrate all the "ghosts for ghosts" and the additionally introduced multipliers, arriving to

$$Z_{TM}{}^{red} = \int \mathcal{D}\Gamma \rho e^{i\int \mathcal{L}} , \tag{21}$$

with

$$\mathcal{L} \sim p\dot{q} - \mathcal{H}_{TM} - A_0\theta - B_{0i}\theta_i - E\Upsilon - E_i\Upsilon_i + \int d^3y \overline{D}_a\{\Upsilon_a, \Theta_b(y)\}D_b(y) , \tag{22}$$

$\mathcal{D}\Gamma = \mathcal{D}p\mathcal{D}q\mathcal{D}D_a\mathcal{D}\overline{D}_a\mathcal{D}E_a\mathcal{D}A_0\mathcal{D}B_{0i}$, and

$$\rho = \delta(D^L_{(i)})\delta(\overline{D}^L_{(i)})\delta(B^L_{(0i)})\delta(E^L_{(i)}) . \tag{23}$$

Also, Υ_a are the gauge fixing conditions defined before. Integrating the remaining fields excepting the p's and q's we arrive to

$$Z^{red}_{TM} = \int \mathcal{D}p\mathcal{D}q det\{\Theta_a, \Upsilon_b\}\delta(\theta)\delta(\theta^T_i)\delta(\Upsilon)\delta(\Upsilon^T_i)e^{i\int(p\dot{q}-\mathcal{H}_{TM})} , \tag{24}$$

where we see that the measure in the path integral corresponds to the right-hand side of (17), as we asserted. Following with (17) and taking care of (16)

$$\begin{aligned} Z^{red}_{TM} &= \int \mathcal{D}p\mathcal{D}q det\{\Phi_A, \Phi_B\}^{\frac{1}{2}}\delta(\Phi_A)e^{i\int(p\dot{q}-\mathcal{H}_{TM})} \\ &= \int \mathcal{D}A_\mu \mathcal{D}B_{\mu\nu}e^{iS^4_P} \qquad . \end{aligned} \tag{25}$$

Then, on cohomological trivial sectors of the base manifold the covariant effective action of S^4_{TM} will be S^4_P, stating that under this condition the latter action can be seen as a gauge fixed version of the former. On general grounds to have a global canonical equivalence, we have to modify S^4_P in order to include the topological sectors originally absent in its space of solutions. This inclusion will modify the partition function by a factor that represents the mentioned sectors. These and other feature can be elucidated considering the master action

$$\begin{aligned} S^4_M = \quad & \int d^4x\big[-\frac{1}{4}b_{\mu\nu}b^{\mu\nu} - \frac{\mu^2}{2}a_\mu a^\mu + \frac{1}{3!}\varepsilon^{\mu\nu\lambda\rho}a_\mu H_{\nu\lambda\rho} + \\ & \qquad + \frac{1}{4}\varepsilon^{\mu\nu\lambda\rho}(b_{\mu\nu} - B_{\mu\nu})F_{\lambda\rho}\big] \qquad . \end{aligned} \tag{26}$$

This action has the same gauge invariances of S^4_{TM} (with $b_{\mu\nu}$ and a_μ transforming homogenously). Its dual field version is

$$S^{4\ \star}_M = \int d^4x\Big[\frac{1}{4}t_{\mu\nu}t^{\mu\nu} - \frac{\mu^2}{2}a_\mu a^\mu + \frac{1}{2}(t^{\mu\nu} - T^{\mu\nu})F_{\mu\nu} + +\frac{1}{2}a_\mu\partial_\nu T^{\nu\mu}\Big] \quad , \tag{27}$$

where $T = {}^\star B$, as before, and $t = {}^\star b$.

From S^4_M we obtain the equations of motion

$$b^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\rho}F_{\lambda\rho} \quad , \tag{28}$$

$$a^\mu = \frac{1}{\mu^2 3!}\varepsilon^{\mu\nu\lambda\rho}H_{\nu\lambda\rho}, \tag{29}$$

$$\varepsilon^{\mu\nu\lambda\rho}\partial_\lambda(A_\rho - a_\rho) = 0 \quad , \tag{30}$$

$$\varepsilon^{\mu\nu\lambda\rho}\partial_\nu(B_{\lambda\rho} - b_{\lambda\rho}) = 0 \quad . \tag{31}$$

Using (28) and (29) in S^4_M, the second order action S^4_{TM} is obtained. By the other side, from (30) we learn that a_μ and A_μ differ by a closed form ω_μ. Also, using (31), an analogous situation occurs between $B_{\mu\nu}$ and $b_{\mu\nu}$ (let the corresponding closed form be $\Omega_{\mu\nu}$). Locally we can set $\omega_\mu = \partial_\mu\lambda$ and $\Omega_{\mu\nu} = \partial_\mu L_\nu - \partial_\nu L_\mu \equiv \mathcal{G}_{\mu\nu}$ and going now into S^4_M we obtain a Stuckelberg form of S^4_P

$$S_{St} = \int d^4x\Big[\frac{1}{4}\varepsilon^{\mu\nu\lambda\rho}B_{\mu\nu}F_{\lambda\rho} - \frac{\mu^2}{2}(A_\mu - \partial_\mu\lambda)(A^\mu - \partial^\mu\lambda) -\frac{1}{4}(B_{\mu\nu} - \mathcal{G}_{\mu\nu})(B^{\mu\nu} - \mathcal{G}^{\mu\nu})\Big] \quad , \tag{32}$$

which is invariant under $\delta A_\mu = \partial_\mu\xi$, $\delta B_{\mu\nu} = \partial_\mu\xi_\nu - \partial_\nu\xi_\mu$, $\delta\lambda = \xi$, $\delta L_\mu = \xi_\mu + \partial_\mu\chi$. The exact forms can be gauged out and we recover S^4_P, stating the local equivalence between the models.

In general the solutions of (30) and (31) are as we stated: $a_\mu = A_\mu - \omega_\mu$ and $b_{\mu\nu} = B_{\mu\nu} - \Omega_{\mu\nu}$. This mantains the homogenity of a_μ and $b_{\mu\nu}$ under gauge transformations. Going to S^4_M, we will obtain the gauge invariant action

$$\widetilde{S}^4_P = \int d^4x\Big[-\frac{1}{4}\varepsilon^{\mu\nu\lambda\rho}\Omega_{\mu\nu}F_{\lambda\rho} + \frac{1}{3!}\varepsilon^{\mu\nu\lambda\rho}(A_\mu - \omega_\mu)H_{\nu\lambda\rho} - -\frac{1}{4}(B_{\mu\nu} - \Omega_{\mu\nu})(B^{\mu\nu} - \Omega^{\mu\nu}) - \frac{\mu^2}{2}(A_\mu - \omega_\mu)(A^\mu - \omega^\mu)\Big] \ . \tag{33}$$

The latter action is global and locally equivalent to S^4_{TM}, and it has incorporated the topological sectors not present, originally, in S^4_P. One important feature of $\widetilde{S}^4_P$ is that ω_μ and $\Omega_{\mu\nu}$ can be taken as independent fields and they will be closed forms dynamically. So $\widetilde{S}^4_P$ is the correct modification to S^4_P in order to obtain a complete correspondence with S^4_{TM}. The gauge invariances of $\widetilde{S}^4_P$ are the ones on S^4_{TM} plus $\delta\omega_\mu = \delta A_\mu$, $\delta\Omega_{\mu\nu} = \delta B_{\mu\nu}$. In a different but equivalent approach we can eliminate A_μ and $B_{\mu\nu}$ in S^4_M with (30) and (31) (in this case $A_\mu = a_\mu + \omega_\mu$, $B_{\mu\nu} = b_{\mu\nu} + \Omega_{\mu\nu}$). Doing so, we arrive to the pair of uncoupled actions

$$\begin{aligned} \widetilde{S}^4_P\Big[a, b, \omega, \Omega\Big] &= S^4_P[f, h] - \frac{1}{2}\int d^4x \varepsilon^{\mu\nu\lambda\rho}\Omega_{\mu\nu}\partial_\lambda\omega_\rho \\ &\equiv S^4_P[a, b] - S^4_{BF}[\omega_1, \Omega_2] \,, \end{aligned} \tag{34}$$

where S^4_{BF} is the part that describes the topological sectors incorporated only in S^4_{TM}, and S^4_P describes the local physical degrees of freedom. Taking into account the substitution we just made and equation (34) we notice that $A_\mu = A^P_\mu + A^{BF}_\mu$, and $B_{\mu\nu} = B^P_{\mu\nu} + B^{BF}_{\mu\nu}$ belong to the space of solutions of S^4_{TM} (this assertion holds even in presence of external sources). The space of gauge inequivalent classical solutions of the BF theory, when the base manifold is $M_4 = R \times \Sigma_3$, is a direct sum of the first and second de Rham cohomology groups on Σ_3, and by Hodge's duality this space is even dimensional [29]. Because of the topological character of the BF theory it will not contribute to the physical spectrum but the long distance behaviour of the solutions of S^4_{TM}, when the field strengths tend to zero asymptotically, will be characterized by the periods of the BF's solutions while all this periods cancel, in this limit, for the Proca theory. Let us illustrate this fact considering S^4_P and S^4_{TM} in presence of a point charge ($J^0 = e\delta^3(\vec{x})$, $J^i = 0$) and a vortex ($J^{0i} = \frac{g}{2}\oint_C dy^i \delta^3(\vec{x} - \vec{y})$,$J^{ij} = 0$). The exterior static solutions are

$$A^{TM}_0 = A^P_0 = -eY(\vec{x}), \tag{35}$$

$$\begin{aligned} A^{TM}_i &= A^P_i + A^{BF}_i = \Big[-g\varepsilon_{ijk}\partial_j \oint_C dy^k Y(\vec{x} - \vec{y})\Big] + \\ &\qquad + \Big[g\varepsilon_{ijk}\partial_j \oint_C dy^k C(\vec{x} - \vec{y}) + \partial_i\lambda\Big], \end{aligned} \tag{36}$$

$$B^{TM}_{0i} = {B_{0i}}^P + B^{BF}_{0i} = -\mu^2 g \oint_C dy^i Y(\vec{x} - \vec{y}) + \partial_i B, \tag{37}$$

$$B^{TM}_{ij} = B^P_{ij} + B^{BF}_{ij} = \Big[e\varepsilon_{ijk}\partial_k Y(\vec{x})\Big] + \Big[-e\varepsilon_{ijk}\partial_k C(\vec{x}) + \partial_i b^t_j - \partial_j b^t_i\Big] \,, \tag{38}$$

where $C(\vec{x}) = [4\pi|\vec{x}|]^{-1}$ and $Y(\vec{x}) = [4\pi|\vec{x}|]^{-1} e^{-\mu^2|\vec{x}|}$ are respectively the Coulomb and Yukawa Green functions ($(-\Delta + \mu^2)Y(\vec{x}) = (-\Delta)C(\vec{x}) = \delta^3(\vec{x})$), and the arbitrariness in λ, B and b_i^t ($\partial_i b_i^t = 0$) due to gauge invariance is shown. These solutions are well defined outside sources and in this region $H^{BF}_{\mu\nu\lambda} = 0$, $F^{BF}_{\mu\nu} = 0$, while for the Proca solutions the field strengths tend to zero asymptotically. If we take an sphere of radius R surrounding the origin we get

$$I_B^{BF} \equiv \oint_{|\vec{x}|=R} B_{ij}^{BF} dx^i \wedge dx^j = 2e \; . \tag{39}$$

This value is independent of the closed surface and is zero when the charge is outside. For the Proca solution

$$I_B^P = -2e(1 + \mu R)e^{-\mu R} \; , \tag{40}$$

and we note that $I_B^P \to 0$ as $R \to +\infty$, so $I_B^{TM} \to I_B^{BF}$ in this limit. I_B^{BF} is the period of the closed 2-form $B = B_{ij} dx^i \wedge dx^j$ and we see, as we stated, that this period labels the TM solutions asymptotically.

For A_i, we have

$$\begin{aligned} I_A^{BF} &= \oint_{C'} dx^i A_i = g\varepsilon_{ijk} \oint_{C'} dx^i \oint_C dy^j \frac{(x^k - y^k)}{|\vec{x} - \vec{y}|^3} \\ &= -\frac{g}{4\pi} \int ds \int ds' (\frac{\partial \hat{u}}{\partial s} \times \frac{\partial \hat{u}}{\partial s'}) \cdot \hat{u} \\ &= -gL(C', C) \; , \end{aligned} \tag{41}$$

where $L(C', C)$ is the linking number of the closed paths C' and C. The unit vector $\hat{u}(s, s')$ is defined by the parametrization of the paths as $\hat{u} = |\vec{R}(s, s')|^{-1} \vec{R}(s, s')$, with $\vec{R}(s, s') = \vec{x}(s') - \vec{y}(s)$. I_A^{BF} corresponds to the period of the closed 1-form $A = A_i dx^i$, and it is a topological invariant. For the Proca solution we will get

$$I_A^P = \frac{g}{4\pi} \int ds \int ds' (\frac{\partial \hat{u}}{\partial s} \times \frac{\partial \hat{u}}{\partial s'}) \cdot \hat{u} (1 + \mu R(s, s')) e^{-\mu R(s,s')} \; . \tag{42}$$

This integral is not a topological invariant and becomes negligible when the paths are, point to point, far apart. So, also in this aspect the TM and Proca solutions have different behaviour.

Now, to end our discussion of the 3+1 models we note that a path integral approach tells us, from (33), that the partition function $\widetilde{Z}_P^4$ is equal to Z_{TM}^4, up to a factor independent of the fields. This is obtained integrating the "omegas". From (34) we obtain that the partition function of S_{TM}^4 and S_P^4 differ by a topological factor

$$Z_{TM}^4 = Z_{BF}^4 Z_P^4 \; . \tag{43}$$

This topological factor, Z^4_{BF}, is proportional to the Ray-Singer analytic torsion of the manifold M_4 [27] [28] [29]. To see this we perform the canonical analysis of S^4_{BF} and note that the ghost for ghost structure is analogous to the one in S^{TM}. To obtain the covariant effective action we make the identifications:

$$D_\mu = (d, D_i), \overline{D}_\mu = (\overline{d}, \overline{D}_i), E_\mu = (\dot{d} - b, E_i) \ , \tag{44}$$

so $\hat{\delta} D_\mu = \partial_\mu d_1$, $\hat{\delta}\overline{D}_\mu = -E_\mu + \partial_\mu d_2$, $\hat{\delta} E_\mu = \partial_\mu d_3$ and $\hat{\delta} B_{\mu\nu} = \partial_\mu D_\mu - \partial_\nu D_\mu$. In the covariant Lorentz gauge, the BRST invariant effective action results to be

$$S^{BF}_{eff} = S_{\mathcal{B}} + S_{\mathcal{F}} \ , \tag{45}$$

where the bosonic part is

$$\begin{aligned} S_{\mathcal{B}} = \int d^4x \Big[\frac{1}{4}\varepsilon^{\mu\nu\lambda\rho} B_{\mu\nu} F_{\lambda\rho} \quad &- \quad E\partial^\mu A_\mu - E^\mu \partial^\nu B_{\mu\nu} - b_3 \partial^\mu E_\mu \\ &+ \quad \overline{d}_1 \partial_\mu \partial_\mu d_1 + \overline{d}_2 (\partial^\mu \partial_\mu d_2 - \partial^\mu E_\mu)\Big] \ , \end{aligned} \tag{46}$$

and the fermionic part is

$$\begin{aligned} S_{\mathcal{F}} = -\int d^4x \Big[b_1 \partial_\mu D_\mu \quad &+ \quad b_2 \partial_\mu \overline{D}_\mu + \overline{C} \partial_\mu \partial_\mu C \\ &+ \quad \overline{d}_3 \partial_\mu \partial_\mu d_3 + \overline{D}_\mu \partial_\nu (\partial^\mu D^\nu - \partial^\nu D_\mu)\Big] \ . \end{aligned} \tag{47}$$

Now, we take S_{eff} on a compact Riemmanian manifold M_4 without boundary where we have the inner product between p-forms $(\omega_p|\gamma_p) = \int_{M_4} \omega_p \wedge \star\gamma_p$ so the adjoint exterior derivative is $\delta_p = (-1)^{np+n+1} \star d \star$. The Laplacian on p-forms is, as usual, $\Delta_p = \delta_{p-1} d + d\delta_p$. On M_4, $S_{\mathcal{B}}$ and $S_{\mathcal{F}}$, take the form

$$\begin{aligned} S_{\mathcal{B}} = \quad & \frac{1}{2}(B| \star dA) - (b|\delta A) - \frac{1}{2}(E|\delta B) - (b_3|\delta E) \\ & +(\overline{d}_1|\Delta_0 d_1) + (\overline{d}_2|\Delta_0 d_2 - \delta E) \end{aligned} \tag{48}$$

$$S_{\mathcal{F}} = \quad -(b_1|\delta D) - (b_2|\delta\overline{D}) - (\overline{C}|\Delta_0 C) - (\overline{d}_3|\Delta_0 d_3) + (\overline{D}|\delta d D) \ , \tag{49}$$

where $D = D_\mu dx^\mu$, $\overline{D} = \overline{D}_\mu dx^\mu$, $E = E_\mu dx^\mu$. Integrating the bosonic fields in the path integral we will get $Z_{\mathcal{B}} = (det\Delta_0)^{-\frac{5}{2}} (det\Delta_1)^{-\frac{1}{2}} (det\Delta_2)^{-\frac{1}{4}}$, up to a field independent factor. Doing first the b's integration in the fermionic part, and then the others we obtain $Z_{\mathcal{F}} = (det\Delta_0)^2 det\Delta_1$, up to an, also, field independent factor. We must observe that up to this point we have assumed the absence of zero modes. This does not contradict our previous arguments because the path integration is made over the coexact pieces of all the fields involved, with their exact pieces gauged fixed. Then

$$Z^4_{BF} = \int [\mathcal{D}h] T^{-\frac{1}{4}}(M_4) \ , \tag{50}$$

where $[\mathcal{D}h]$ indicates that an integration over the zero-modes remains to be done, and $T(M_4)$ represents the Ray-Singer analitical torsion of M_4

$$T(M_4) = (det\Delta_0)^2 (det\Delta_1)^{-2} det\Delta_2 \ , \tag{51}$$

with the determinants computed *via* ζ-function regularization [27], so only non-zero eigenvalues contribute. When the manifold is cohomologically trivial (so there are no zero-modes) $det\Delta_2 = (det\Delta_1)^2 (det\Delta_0)^{-2}$ (Proposition 4 in [28]), then $T(M_4) = 1$ and $Z^4_{BF} = 1$, ensuring the complete equivalence between S^4_{TM} and S^4_P. In general, the zero mode integration will give a factor that is also a topological invariant. For an even dimensional compact manifold without boundary the Ray-Singer torsion is trivial, but fron (50) we observe that $Z_{BF} \neq 1$. The integration over the zero modes must be kept in order to have an appropiate path integral measure for computing expectation values [30] [29]. This integration runs over a graded sum of cohomology groups due to the alternating parity of the fields involved [30] [28]. For odd dimensional compact manifolds without boundary the Ray-Singer is in general non-trivial, even in the absence of zero modes.

Finally, we quote that all these results are generalized trivially to d+1 dimensions. The corresponding models are written as (7) and (8) or equivalently in terms of the $(d-1)$-form B

$$S^{d+1}_P = \int_{M_{d+1}} \Big[\frac{1}{2} B_{d-1} \wedge F + \frac{1}{8} B_{d-1} \wedge {}^\star B_{d-1} + \frac{\mu^2}{2} A \wedge {}^\star A \Big] \ , \tag{52}$$

and

$$S^{d+1}_{TM} = \int_{M_{d+1}} \Big[\frac{1}{8\mu^2} H \wedge {}^\star H + \frac{1}{2} F \wedge {}^\star F - \frac{1}{2} B_{d-1} \wedge F \Big] \ , \tag{53}$$

where $H = dB$ and $F = dA$. These actions can be extended to d=2. In the latter case each one of the corresponding models describe two massive spin 1 excitations as the Proca model in 2+1 dimensions. For $d \geq 3$ the connection between (52) and (53) is analogous to that of the 3+1 analized models:

- Both models describe the same physical spectrum as the Proca model, which is described by d independent physical degrees of freedom.

- S_P^{d+1} is locally a gauge fixed version of S_{TM}^{d+1} .

- S_{TM}^{d+1} has a topological sector in its space of solutions not present in the former. This topological sector corresponds to the space of classical solutions of the BF model (with Lagrangian density $\mathcal{L}_{BF} = B \wedge dA$), and is responsible of the different long distance behaviour of the physical models, where the field strengths tend to zero asymptotically.

- The presence of the topological sector appears as a topological factor in their partition functions: $Z_{TM}^{d+1} = Z_{BF}^{d+1} Z_P^{d+1}$. In D dimensions the partition function for the BF model becomes [28]

$$Z_{BF}^D = \begin{cases} T(M_D)^{-1} & \text{for D odd} \\ T(M_D)^{\frac{3-D}{D}} & \text{for D even} \end{cases} , \tag{54}$$

  where $T(M_D)$ is the Ray-Singer analitical torsion of the base manifold, and the integration over zero modes remains to be done.

- On cohomologically trivial base manifolds both free models are identical, and it can be said on general grounds that the BF solutions label Proca formulations on sectors of the manifold with trivial structure.

- There is a master action that connects both models. It is

$$S_M^{d+1} = \int_{M_{d+1}} \Big[\frac{1}{8} b_{d-1} \wedge {}^\star b_{d-1} + \frac{\mu^2}{2} a \wedge {}^\star a - \frac{1}{4} a \wedge H + \frac{1}{2} (b_{d-1} - B_{d-1}) \wedge F \Big] . \tag{55}$$

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